THE HOPF INVARIANT AND SIMPLEX STRAIGHTENING

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ABSTRACT. Let M be a closed oriented 3-manifold which can be triangulated with N simplices. We prove that any map from M to a genus 2 surface has Hopf invariant at most C^N. Let X be a closed oriented hyperbolic 3-manifold with injectivity radius less than ϵ at one point. If there is a degree non-zero map from M to X, then we prove that ϵ is at least C^{-N}.

In the 1970's, Thurston invented simplex straightening. Milnor and Thurston used simplex straightening to bound the degrees of maps to hyperbolic manifolds. In this paper, we extend their method to estimate the Hopf invariant instead of the degree.

Here is a degree estimate that Milnor and Thurston proved.

Theorem. *(Milnor, Thurston) Let M be a closed oriented n-manifold that can be triangulated by N simplices, and let X be a closed oriented hyperbolic n-manifold with volume V. If f is any continuous map from M to X, the norm of the degree of f is bounded by CN/V.*

We prove a similar estimate for the Hopf invariant.

Theorem 1. *Let M be a closed oriented 3-manifold which can be triangulated with N simplices. Let f be a continuous map from M to a closed oriented surface of genus 2. If the Hopf invariant of f is defined, then its norm is less than C^N.*

(The Hopf invariant can be defined for a map from an oriented 3-manifold to an oriented surface provided that the pullback of the fundamental cohomology class of the surface is zero. See Section 1 for more details.)

Our bound for the Hopf invariant grows exponentially in N. We will construct examples where the Hopf invariant is greater than $(1+c)^N$ for a universal constant $c > 0$.

Using estimates related to Theorem 1, we give a new proof of a theorem of Soma:

Theorem. *(Soma) If M is a closed oriented 3-manifold, then there are only finitely many closed oriented hyperbolic 3-manifolds admitting maps of non-zero degree from M.*

Soma's theorem follows from the following quantitative estimate:

Theorem 2. *Let M be a closed oriented 3-manifold that can be triangulated by N simplices. Let X be a closed oriented hyperbolic 3-manifold with injectivity radius ϵ. If there is a map of non-zero degree from M to X, then ϵ is at least C^{-N}.*

Roughly speaking, Theorem 2 says that a closed oriented hyperbolic manifold with small injectivity radius at one point must be topologically complicated.

Trying to understand Soma's theorem was the main motivation for the work in this paper. Here is some context for Soma's theorem. In dimension greater than 3,

the analogue of Soma's theorem is true, and the proof is much easier. In dimension greater than 3, there are only finitely many closed hyperbolic manifolds with volume less than a given bound. Applying the theorem of Milnor and Thurston, it follows immediately that a given n-manifold M admits maps of non-zero degree to only finitely many hyperbolic n-manifolds. This proof does not work in dimension 3, because there are infinitely many closed hyperbolic 3-manifolds with volume less than a given bound.

In dimension greater than 3, the analogue of Soma's theorem also holds for non-orientable n-manifolds. The proof is essentially the same as in the orientable case. In [1], Boileau and Wang proved that the analogue of Soma's theorem is false for non-orientable 3-manifolds. Our proof of Theorem 2 requires X to be orientable because we use the Hopf invariant of maps from X to the 2-sphere. (If X is non-orientable, then the Hopf invariant is only defined modulo 2, and our arguments break down.)

The proof of Theorem 1 has two ingredients. First, we use the simplex straightening method to homotope an arbitrary continuous map to a map with a bounded Lipschitz constant. Second, we use a method of Gromov (see [3]) to bound the Hopf invariant of a map in terms of its Lipschitz constant.

The proof of Theorem 2 also begins with simplex straightening. An arbitrary map $f : M \to X$ can be straightened to a map with Lipschitz constant 1. There are many obstructions to finding a degree non-zero map with controlled Lipschitz constant. The total volume gives the simplest obstruction, and the volume estimate leads to Milnor-Thurston degree estimate. But there are other obstructions to finding a 1-Lipschitz map of non-zero degree. In this paper, we explore a different obstruction, related to a new Riemannian invariant called the Hopf size.

The Hopf size of a closed oriented 3-manifold (M, g) measures the largest Hopf invariant of an L-Lipschitz map to the unit 2-sphere, with an appropriate normalization depending on L.

$$\text{Hopf Size } (M^3, g) := \sup_{L} \ \sup_{\text{Lip } (F) \leq L} |\text{Hopf } (F)| L^{-4}.$$

We prove that the Hopf size of a closed oriented hyperbolic 3-manifold X controls the degrees of maps to X.

Proposition 1. *Let M be a closed oriented 3-manifold triangulated by N simplices. Let X be a closed oriented hyperbolic 3-manifold. Let $f : M \to X$ be a continuous map.*

$$|Degree \ (f)| \leq C^N / Hopf \ Size \ (X).$$

Proposition 1 is a variation of the Milnor-Thurston degree estimate, using the Hopf size of X instead of the volume of X. Proposition 1 gives new information not covered by the Milnor-Thurston estimate because a hyperbolic 3-manifold may have volume less than 100 and arbitrarily large Hopf size.

To prove Theorem 2, we perform a classification of closed oriented hyperbolic 3-manifolds with small injectivity radius. Most of these hyperbolic manifolds have large Hopf size. The rest of them are topologically complicated in a different way, such as having a torsion subgroup of H_1 with large order.

In Section 1, we review simplex straightening, the Hopf invariant, and Gromov's estimate. In Section 2, we prove Theorem 1. In Section 3, we introduce the Hopf

size and prove Proposition 1. In Section 4, we prove Theorem 2. In Section 5 we give an example to show that the exponential growth in Theorem 1 is real. In Section 6 we give a brief exposition of the Boileau-Wang theorem for maps to non-orientable 3-manifolds.

Notation: We use $|y|$ for the mass of a real chain y in a Riemannian manifold. We use the constant C to denote a positive constant that may change from line to line.

This paper is based on a section of my thesis, and I would like to thank my advisor Tom Mrowka for his help and support.

1. Background

In this section we review simplex straightening, the Hopf invariant, and Gromov's Hopf estimate.

First we describe simplex straightening. By a hyperbolic k-simplex, we mean a (finite) k-dimensional geodesic simplex in hyperbolic k-space. By Lobachevsky's theorem, the volume of a hyperbolic k-simplex is bounded by $C(k)$ for each $k \geq 2$. A hyperbolic simplicial complex is a simplicial complex where each simplex is given the metric of a hyperbolic simplex, and the metrics agree in the obvious way.

Simplex straightening. *(Thurston) Suppose that M is a simplicial complex and X is a complete hyperbolic manifold, and that F_0 is a continuous map from M to X. Then we can choose hyperbolic metrics on each simplex of M, and we can homotope F_0 to a new map F in such a way that F is a local isometry. In particular, the Lipschitz constant of F is 1.*

Next we review the Hopf invariant. Suppose that $F : M \to \Sigma$ is a map from an oriented 3-manifold M to an oriented surface Σ. Let ω denote the fundamental cohomology class in $H^2(\Sigma, \mathbb{R})$. We will define the Hopf invariant of F provided that $F^*(\omega) = 0$.

Let q be a generic point in Σ. Let $y = f^{-1}(q) \subset M$. The fiber y is an oriented 1-manifold. Since $F^*(\omega) = 0$, it follows by Poincare duality that the homology class of y vanishes in $H_1(M, \mathbb{R})$. Let z be a real 2-chain with $\partial y = z$. If we restrict the map F to z, we get a map from $(z, \partial z)$ to (Σ, q). This map has a well-defined degree, which is equal to the Hopf invariant of F. (The push-forward of z is a real 2-cycle in Σ, and the Hopf invariant of F is the homology class of this cycle.)

By standard arguments from differential topology, the Hopf invariant does not depend on the choice of z or the choice of q, and it is a homotopy invariant of the map F. See [4] for more background.

Now we turn to Gromov's geometric estimate for the Hopf invariant. The question is as follows. Suppose that a map $F : (M^3, g) \to (\Sigma^2, h)$ has Lipschitz constant L. In terms of L and geometric information about the domain and range, how can we bound the Hopf invariant of F?

(To get a feeling for this problem, let's quickly consider the simpler case of bounding the degree of a map $F : (M^n, g) \to (N^n, h)$. The norm of the degree is at most $L^n Vol(M)/Vol(N)$. For the Hopf invariant, we cannot get a bound knowing only the Lipschitz constant L, the volume of M, and the area of Σ. We need some more refined geometric information.)

Gromov's bound depends on understanding isoperimetric inequalities in (M^3, g). We define $Iso(M, g)$ as the smallest number so that every null-homologous real 1-cycle y in (M, g) bounds a 2-chain of mass at most $Iso(M, g)|y|$. Gromov bounded

the Hopf invariant of F in terms of the Lipschitz constant L, the volume of (M, g), the area of (Σ, h), and this isoperimetric constant:

Gromov's Hopf invariant estimate. *([3], see also [4]) Suppose that $F : (M^3, g) \to (\Sigma^2, h)$ has Lipschitz constant L. Then the Hopf invariant of F has norm at most $Iso(M, g)Vol(M, g)Area(\Sigma, h)^{-2}L^4$.*

Proof. (For a more detailed proof, see [4].)

By the coara formula, we can choose a generic $q \in \Sigma$ so that the length of y is at most $L^2Vol(M)/Area(\Sigma)$. By assumption, we can choose z with mass at most $Iso(M)Length(y) \le Iso(M)L^2Vol(M)Area(\Sigma)^{-1}$. The map from z to Σ has Lipschitz constant L. So its degree has norm at most $L^2|z|Area(\Sigma)^{-1} \le L^4Iso(M)Vol(M)Area(\Sigma)^{-2}$. $\qquad\square$

For example, the Hopf invariant of an L-Lipschitz map from the unit 3-sphere to the unit 2-sphere has norm at most CL^4. The exponent 4 here is sharp. For $L > 2$, there are maps from the unit 3-sphere to the unit 2-sphere with Lipschitz constant L and Hopf invariant at least cL^4. For more information, see [4] or Chapter 7 of [2].

In order to apply Gromov's inequality, one has to estimate $Iso(M, g)$ for the Riemannian metric at hand. The isoperimetric constant $Iso(M, g)$ is not as easy to compute/estimate as a volume or a Lipschitz constant. For example, if (M, g) is a 3-dimensional ellipse, then $Iso(M, g)$ turns out to be roughly the second longest principal axis of the ellipse, as proven in [4]. Even this special case takes some work.

2. Estimates for the Hopf invariant

In this section, we prove Theorem 1.

Theorem 1. *Let M be a closed oriented 3-manifold which can be triangulated with N simplices. Let Σ be a closed surface of genus 2 with fundamental cohomology class $\omega \in H^2(\Sigma, \mathbb{R})$. Let f be a continuous map $f : M \to \Sigma$ with $f^*(\omega) = 0$. Then the Hopf invariant of f has norm at most C^N.*

Proof. Let Σ be the surface of genus 2. We equip it with a hyperbolic metric of area 4π. Next we apply simplex straightening to the map f. We get a new map F, homotopic to f, and we give M the structure of a hyperbolic simplicial complex. We can think of this structure as a Riemannian metric g on M. (The metric is continuous but not smooth across the lower-dimensional faces of the triangulation. This minor lack of regularity is not important.) Simplex straightening tells us that the map $F : (M^3, g) \to (\Sigma^2, hyp)$ has Lipschitz constant 1.

We will estimate the Hopf invariant of F using a variant of Gromov's method. The technical difficulty that we face is that the space of hyperbolic simplices is not compact. Hence the space of possible metrics g on M is not compact. We will prove a uniform estimate for the Hopf invariant of F.

The volume $Vol(M, g)$ is uniformly bounded by CN. If we had a uniform bound on $Iso(M, g)$, then we could apply Gromov's Hopf invariant estimate and get a bound on the Hopf invariant of f. But I don't know any uniform bound for $Iso(M, g)$. I suspect there is no uniform bound for $Iso(M, g)$. We will get around this problem by using a small variation of Gromov's idea. The key estimate is the following isoperimetric inequality.

Filling Lemma. *Suppose that (M^3, g) is a hyperbolic simplicial complex with N simplices. Suppose that y is an integral 1-cycle in M^3, and that y is null-homologous in $H_1(M, \mathbb{R})$. Suppose that the mass of y is at most B. Also, suppose that y is transverse to the 2-skeleton of M, and that y intersects the 2-skeleton in at most B points.*

Then y bounds a real 2-chain z with mass at most $C^N B$.

Using the Filling Lemma, we finish the proof of the theorem.

By Lobachevsky's theorem about the volumes of hyperbolic simplices, we know that the volume of (M^3, g) is at most CN, and the area of the 2-skeleton is also at most CN.

By the coarea formula, we can choose a generic point $q \in \Sigma$ so that $y = F^{-1}(q)$ has length at most CN and the number of intersections of y with the 2-skeleton is at most CN. By Poincaré duality y is null-homologous. By the Filling Lemma, we can choose a real 2-chain z with mass at most NC^N with boundary y. Since F has Lipschitz constant 1, the degree of the map from z to Σ has norm at most CNC^N. Adjusting C, we see that the Hopf invariant of F is at most C^N. □

Now we turn to the proof of the Filling Lemma. The main ingredient is a variant of the Federer-Fleming deformation theorem.

Deformation Lemma. *Suppose that (M^3, g) is a hyperbolic simplicial complex with N simplices. Suppose that y is an integral 1-cycle in M^3. Suppose that the length of y is at most B and that the number of intersections of y with the 2-skeleton of M is also at most B.*

Then y is homologous to a 1-cycle y' in the 1-skeleton of M. If e_i denote the (oriented) edges of M, then $y' = \sum_i c_i e_i$ with $c_i \in \mathbb{Z}$. Moreover, y' obeys the estimate

$$\sum_i |c_i| < CB$$

and y' is homologous to y by a 2-chain of area at most CB.

Proof. We consider the intersection of y with each open 3-simplex in M. Topologically, each component of the intersection is either a segment with endpoints in the boundary of a 3-simplex or else a closed curve in the simplex.

Step 1. (Straightening y.) We construct a cycle y_2 simpex by simplex in the following way. For each topological segment in $y \cap \Delta^3$, we put the geodesic segment in Δ^3 with the same endpoints.

Now y_2 still has length at most B and it still intersects the 2-skeleton of M in at most B points. Moreover, the standard isoperimetric inequality in hyperbolic space implies that y is homologous to y_2 by a 2-chain of area at most CB.

Step 2. (Pushing y_2 to the 2-skeleton.) We construct a 1-cycle y_3 from y_2 as follows. For each simplex Δ^3, and each segment of y_2 in Δ^3, we choose a path in the boundary of Δ^3 connecting the endpoints of the segment. The path in the boundary consists of two geodesic segments.

The cycle y_3 may be much longer than y_2. It lies in the 2-skeleton of M, and it consists of at most CB geodesic segments. Each of these segments has one endpoint in the 1-skeleton of M.

The homology from y_2 to y_3 consists of at most CB hyperbolic triangles, and so it has area at most CB.

Step 3. (Pushing y_3 to the 1-skeleton.) The cycle y_3 is a union of geodesic segments. Each segment has one endpoint in the 1-skeleton of M. Topologically, y_3 is a union of circles. Each circle is a chain of geodesic segments in the 2-skeleton of M. Every other endpoint in the chain of segments lies in the 1-skeleton of M.

If p is an endpoint in the chain of segments lying in the interior of a 2-face of M, the choose a vertex $v(p)$ in the boundary of this 2-face. Next, suppose that $\sigma \subset y_3$ is a geodesic segment from q to p, where q lies in the 1-skeleton of M. We replace σ by a path from q to $v(p)$, lying in the boundary of the 2-face containing p. This path consists of at most two geodesic segments each in the 1-skeleton of M.

We let y_4 be the sum of the new chains. We see that y_4 is a union of at most CB geodesic segments, where each segment lies in one edge of the 1-skeleton. Hence $y_4 = \sum a_i e_i$ where e_i are the edges of the 1-skeleton and $\sum |a_i| \le CB$.

Next we bound the area of the homology from y_3 to y_4. Let σ' be the other segment touching p, and suppose that σ' goes from p to q'. Adding σ, σ', and their replacements, we get a 1-cycle in a single 2-simpex of M, consisting of at most 6 geodesic segments. It can be filled by a chain consisting of at most 4 geodesic triangles. By Lobachevsky's theorem, the filling has area at most C. Since the total number of segments in y_3 was at most CB, the homology from y_3 to y_4 has mass at most CB.   □

Now we fill this 1-cycle using linear algebra.

Let e_i denote the edges of M and f_j denote the faces of M. We assign them orientations in an arbitrary way. Then the simplicial boundary map is given by a matrix β_{ij}, where $\partial f_j = \sum \beta_{ij} e_i$. The entries of β_{ij} are each ± 1 or 0. The number of rows and the number of columns of the matrix are both at most CN. Because each 2-simplex has only three 1-simplices in its boundary, each column of β has at most three non-zero entries.

Linear algebra lemma. *If $y' = \sum c_i e_i$ is a null-homologous real 1-cycle, then we can find a real 2-chain $z = \sum d_j f_j$ with $\partial z = y'$, and $\sum_j |d_j| \le C^N \sum_i |c_i|$.*

Proof. Recall that the Hilbert-Schmidt norm of a matrix with entries β_{ij} is defined to be $(\sum |\beta_{ij}|^2)^{1/2}$. The Hilbert-Schmidt norm of the boundary matrix β is bounded by $(CN)^{1/2}$.

Let r be the rank of the boundary matrix β. We can choose orthogonal matrices O_1 and O_2 so that $O_1 \beta O_2$ is zero outside of the top left-hand $r \times r$ sub-matrix. We define β' to be this $r \times r$ sub-matrix. Because the rank of β is r, one of the $r \times r$ sub-determinants of β is not zero. Because the entries of β are all integers, this sub-determinant is an integer, and so its norm is at least 1. The norm of the determinant of β' is at least as large as the norm of any $r \times r$ sub-determinant of β. Therefore, the determinant of β' has norm at least 1. The Hilbert-Schmidt norm of β' is equal to that of β, which is less than $(CN)^{1/2}$. Any $(r-1) \times (r-1)$ sub-matrix of β' has smaller Hilbert-Schmidt norm. The determinant of an arbitrary matrix a_{ij} is bounded by $\Pi_i (\sum_j |a_{ij}|^2)^{1/2}$. It follows from this estimate and our bound on the Hilbert-Schmidt norm that every $(r-1) \times (r-1)$ sub-matrix of β' has determinant bounded by C^N. Since the determinant of β' has norm at least 1, the norm of every entry in the inverse of β' is bounded by C^N. So the norm of the inverse of β' is bounded by CNC^N.

Therefore, the exact 1-chain $\sum c_i e_i$ is equal to the boundary of a 2-chain $\sum d_j f_j$, with $\sum |d_j|$ bounded by $CN^2 C^N \sum |c_i|$. After increasing the constant C, we can say that $\sum |d_j|$ is bounded by $C^N \sum |c_i|$. □

Assembling all of our steps, we have produced a real 2-chain z with $\partial z = y$ and mass at most $C^N B$. This finishes the proof of the Filling Lemma and hence the proof of Theorem 1.

3. The Hopf size

In this section we define a notion of size based on the Hopf invariant.

Let (M^3, g) be a closed oriented Riemannian 3-manifold. We consider maps F from (M^3, g) to the unit 2-sphere. Let $\omega \in H^2(S^2, \mathbb{R})$ be the fundamental cohomology class of the 2-sphere. We recall from Section 1 that the Hopf invariant of F is defined whenever $F^*(\omega) = 0$ in $H^2(M, \mathbb{R})$. The Hopf size of M is defined as follows.

$$\text{Hopf Size } (M^3, g) := \sup_L \sup_{\text{Lip } (F) \leq L} |\text{Hopf } (F)| L^{-4}.$$

Here the second sup is taken over all maps F from (M^3, g) to the unit 2-sphere with Lipschitz constant at most L and with $F^*(\omega) = 0$. The Hopf size is a finite positive number. One can give an upper bound by using Gromov's Hopf invariant estimate from Section 1 - this shows that the Hopf size is finite. On the other hand, there are examples of maps from the unit 3-sphere to the unit 2-sphere with Hopf invariant at least cL^4, which shows that the Hopf invariant of the unit 3-sphere is positive. But any closed oriented 3-manifold admits a Lipschitz degree 1 map to the unit 3-sphere, and so it too has positive Hopf size, by Proposition 3.2 below.

The idea of Hopf size is essentially due to Gromov. In [3], Gromov gave a related definition for the 'area' of a Riemannian 3-manifold. Our definition is a small variation on his, adapted to the topology problems in this paper.

In [2], Chapter 2, Gromov studied the analogous definition for the degree. If (M^n, g) is a closed orientable manifold, then

$$\text{Degree Size } (M^n, g) := \sup_L \sup_{\text{Lip } (F) \leq L} |\text{Degree } (F)| L^{-n}.$$

Here the second sup is taken over all maps F from (M^n, g) to the unit n-sphere with Lipschitz constant at most L. Gromov proved that the degree size of (M^n, g) agrees with the volume of (M^n, g) up to a constant factor $C(n)$. So the degree size essentially is the volume. The Hopf size is analogous to the degree size but subtler. We will see below that it is not proportional to the volume.

The methods in the last section bound the Hopf size of a manifold triangulated into hyperbolic simplices.

Proposition 3.1. *Let (M^3, g) be a closed oriented manifold triangulated into N simplices, so that the restriction of g to each simplex is the metric of a hyperbolic simplex.*

$$\textit{Hopf Size}(M^3, g) \leq C^N.$$

Proof. Consider a map F from (M^3, g) to the unit 2-sphere with Lipschitz constant L and with $F^*(\omega) = 0$. By the coarea formula, we can choose $q \in S^2$ so that $F^{-1}(q)$ has length at most CNL^2 and meets the 2-skeleton of M^3 in at most CNL^2 points. By the Filling Lemma from Section 2, $F^{-1}(q)$ bounds a real 2-chain with mass at most $C^N CNL^2$. But then the push-forward of z to S^2 has mass at most $C^N CNL^4$. After redefining the constant C, the Hopf invariant of F has norm at most $C^N L^4$. $\qquad\square$

The Hopf size can control the degrees of Lipschitz maps between Riemannian manifolds.

Proposition 3.2. *Suppose that (M, g) and (M', g') are closed oriented Riemannian 3-manifolds, and that F is a map from M to M' with Lipschitz constant L.*

$$|Degree\ (f)| \le L^4 \frac{Hopf\ Size\ (M, g)}{Hopf\ Size\ (M', g')}.$$

Proof. By definition, we can find a map h from (M', g') to the unit 2-sphere with $|Hopf(h)|Lip(h)^{-4}$ as close as we like to the Hopf size of (M', g'). Next we consider the composition $h \circ f$ from (M, g) to the unit 2-sphere. It has Lipschitz constant at most $L \cdot Lip(h)$ and Hopf invariant equal to $deg(f)Hopf(h)$. $\qquad\square$

Now using simplex straightening, the degree of a smooth map from M to a hyperbolic 3-manifold X can be bounded in terms of the Hopf size of X.

Proposition 3.3. *Let M be a closed oriented 3-manifold triangulated by N simplices. Let X be a closed oriented hyperbolic 3-manifold. Let $f : M \to X$ be a continuous map.*

$$|Degree\ (f)| \le C^N / Hopf\ Size\ (X).$$

Proof. By simplex straightening, we can find a hyperbolic triangulation g for M and a map F homotopic to f with Lipschitz constant 1. By Proposition 3.1., the Hopf size of (M, g) is at most C^N. By Proposition 3.2., the result follows. $\qquad\square$

4. THE PROOF OF THEOREM 2

First we recall the statement of Theorem 2.

Theorem 2. *Let M be a closed oriented 3-manifold that can be triangulated by N simplices. Let X be a closed oriented hyperbolic 3-manifold with injectivity radius ϵ. If there is a map of non-zero degree from M to X, then ϵ is at least C^{-N}.*

The theorem basically says that a hyperbolic 3-manifold with small injectivity radius is topologically complicated. We will show that a closed oriented hyperbolic 3-manifold with small injectivity radius is large / complicated in one of four ways. First, it may have a large volume. Second, it may have a large Hopf size. Third, there may be a torsion element in $H_1(X, \mathbb{Z})$ with a large order. Fourth, it may require surfaces of large genus to span $H_2(X, \mathbb{Z})$. Any of these four features allow us to bound the degree of a map from M to X in terms of the number of simplices of M.

An important example is the case of hyperbolic Dehn fillings of a finite-volume oriented hyperbolic manifold with a single cusp. For motivation, we explain what happens in this case.

Let X_0 be a finite-volume oriented hyperbolic manifold with a single cusp. Let a and b be a basis for the homology of the boundary torus of X_0, which we are going to Dehn fill. Choose the basis so that the homology class a bounds in X_0, and so that the intersection number of a and b is 1. Let $X(m,n)$ be the manifold formed by Dehn filling the boundary torus along the curve homologous to $ma + nb$, for relatively prime numbers m and n. According to Thurston's theory, the manifold $X(m,n)$ admits a hyperbolic structure for all but finitely many choices of (m,n). These hyperbolic manifolds have uniformly bounded volume. They contain a short core geodesic with length on the order of $(m^2 + n^2)^{-1}$. When $n = 0$, this core geodesic is non-torsion in $H_1(X(m,n), \mathbb{Z})$. Otherwise, the core geodesic is torsion with order n. If n is large, then $X(m,n)$ is topologically complicated because of this high-order torsion element.

Our new idea concerns what happens when n is small but m is large. In this case, we will prove that there is a map from $X(m,n)$ to the unit 2-sphere with Lipschitz constant L and with Hopf invariant of norm at least $\sim (m/n)L^4$. Hence $X(m,n)$ has Hopf size on the order of $|m/n|$.

The reason is that uniformly in m and n, the manifold $X(m,n)$ contains two disjoint thick tubes homologous to b, which lie in the thick part of X_0. Neither tube is null-homologous in X_0, so their linking number is not defined in X_0. But in $X(m,n)$, the tubes are (rationally) null-homologous, and a short calculation shows that their linking number is equal to $\pm(m/n)$. Linking numbers of tubes are closely related to the Hopf invariant, and using these tubes, we can construct a map to the unit 2-sphere with Lipschitz constant bounded independent of m, n and with Hopf invariant at least $|m/n|$.

Now we turn to the formal argument in the general case. Let X be a closed oriented hyperbolic 3-manifold with injectivity radius ϵ. Then X contains a closed embedded geodesic γ of length on the order of ϵ. If γ is torsion in homology with a fairly small order, then we will prove that either X has large Hopf size or else X has large volume. This estimate is the main idea in the proof of Theorem 2.

Proposition 4.1. *Let X be a closed oriented hyperbolic 3-manifold with a closed geodesic γ of length ϵ which is torsion in homology. Then either the volume of X is at least $c\epsilon^{-1/6}$, or the order of γ is at least $c\epsilon^{-1/6}$, or the Hopf size of X is at least $c\epsilon^{-1/4}$.*

Proof. Let T be the Margulis tube around γ. (The thin part of X is defined to be the subset of X where the injectivity radius is less than a certain constant, and the Margulis tube is the connected component of the thin region containing γ.) The universal cover of γ is a geodesic in hyperbolic 3-space. We can parameterize hyperbolic space by the upper half-space model so that the universal cover of γ is equal to the vertical line through the origin (i.e. the line $x = 0, y = 0, z > 0$). The group of covering transformations of X includes a loxodromic isometry that fixes this vertical line. This isometry is given by multiplying the three coordinates by a constant on the order of $(1 + \epsilon)$, and rotating in the (x,y)-plane by an angle θ. Consider the quotient of hyperbolic 3-space by this isometry, and let $\tilde{T}$ be the Margulis tube of this quotient around the core geodesic. Taking the quotient by the other covering transformations of X gives a map from $\tilde{T}$ into M, which takes the core geodesic of $\tilde{T}$ onto γ. According to the Margulis Lemma, the map is an embedding of $\tilde{T}$ into the Margulis tube around γ.

The Margulis tube $\tilde{T}$ has a simple form. For some number R depending on ϵ and θ, the tube $\tilde{T}$ is equal to the quotient of the region $x^2 + y^2 < z^2 R^2$ by the action of the loxodromic isometry corresponding to γ. A fundamental region for this action is given by the intersection of the region above with the region $1 \le z \le 1 + \epsilon$. The boundary of the fundamental region includes two disks, the first given by $z = 1$ and $x^2 + y^2 \le R^2$, and the second given by $z = 1 + \epsilon$ and $x^2 + y^2 \le (1 + \epsilon)^2 R^2$. The loxodromic isometry corresponding to γ takes the first disk onto the second disk. In several constructions, we will use the radial curves in $\tilde{T}$, which are the straight rays through the origin in the Euclidean metric on the (x,y,z)-space. These radial lines are not geodesics in hyperbolic space, but they are still useful in our proof.

We will give two estimates for R in terms of ϵ and θ. The first estimate says that if ϵ is small, then R is large. More precisely, the radius R is at least $c\epsilon^{-1/2}$. Beginning at a point p on the edge of $\tilde{T}$, we can follow a radial curve with length $N\epsilon R$ going N times around the edge of $\tilde{T}$. This curve hits the circle $z = 1$, $x^2 + y^2 = R^2$ in N points. Connecting the closest two of these N points by an arc of the circle, we get a homotopically non-trivial closed curve with length less than $N\epsilon R + R/N$. Since this curve lies on the edge of the Margulis tube, its length is greater than a constant on the order 1. We can make this construction for any number N. In particular, if $N = \epsilon^{-1/2}$, we get the inequality $\epsilon^{1/2} R \ge c$. This proves our lower bound $R > c\epsilon^{-1/2}$. On the other hand, we can assume that R is not too big. The volume of $\tilde{T}$ is roughly ϵR^2. If the volume of X is at least $c\epsilon^{-1/6}$ then we are done, so we may assume that $R < C\epsilon^{-7/12}$.

Our second estimate says that if $\theta/(2\pi)$ is well-approximated by rational numbers, then R is large. Again, we begin at a point at radius R from the the center of the horosphere $z = 1$, and we trace a radial curve that goes vertically q times around the tube, and then connect the endpoint of this curve to the starting point within the horosphere $z = 1$. The total length of this curve is roughly $q\epsilon R + |q\theta/(2\pi) - p|R$, where p is the integer that makes this expression smallest. Since this curve is homotopically non-trivial and lies on the edge of the Margulis tube, it must have length at least on the order of 1. Therefore, $|\theta/(2\pi) - p/q| > c(1/q)(1/R) - \epsilon$.

We choose generators for the homology of the boundary of $\tilde{T}$, given by l, the longitude, and m the meridian. (The choice of longitude is related to the choice of θ in the following way. Take a radial curve on the edge of our fundamental domain, going once around the tube from $z = 1$ to $z = 1 + \epsilon$. This line connects to a point on the base horosphere $z = 1$. From that point, follow an arc of the circle with directed length $-\theta$ to the initial point of the radial curve. The resulting closed curve is homologous to the longitude. We orient γ so that the z coordinate increases as we go along γ. We orient the longitude to be homologous to γ and we orient the meridian so that it has linking number 1 with γ.) Let $pm + ql$ be the homology class of a primitive curve which bounds in $X - T$. (The numbers p and q are relatively prime integers with $q \ge 0$. Since the proposition assumes that γ is torsion in homology, q is not zero. The number q is the order of γ in $H_1(X)$.) If q is at least $c\epsilon^{-1/6}$ we are done, so we will assume that q is smaller than $c\epsilon^{-1/6}$. The inequality at the end of the last paragraph tells us that $|\theta/(2\pi) - p/q| > c\epsilon^{3/4} - \epsilon$. Our proposition is only interesting for small values of ϵ, so we may assume that $c\epsilon^{3/4}$ dominates ϵ, giving us the inequality $|\theta/(2\pi) - p/q| > c\epsilon^{3/4}$.

Finally, we estimate the Hopf size of X. We claim that the Hopf size of X is at least $c|\theta/(2\pi) - p/q|R^2$, which is greater than $c\epsilon^{-1/4}$. To see this, we build

a collection of thin tubes in $\tilde{T} - \gamma$. The tubes are easiest to construct in case $\theta/(2\pi)$ happens to be a rational number M/N. For the time being we make this assumption. Take a radial line in $\tilde{T}$ and follow it N times around the tube $\tilde{T}$ until it comes back to its initial point making a closed circle. Then take a small neighborhood of this circle, also consisting of a union of radial lines. We assume that the tube around our first radial line meets the horosphere $z = 1$ in a region of the following form, using polar coordinates: $|\theta - \theta_0| < \delta$, and $|r - r_0| < r_0\delta$, for some small number δ. (We assume r_0 is bigger than 1, to avoid some unimportant technicalities.) The entire tube meets the horosphere $z = 1$ in N disjoint regions of this kind. This tube is bilipshitz to the euclidean tube $S^1(Nr\epsilon) \times D^2(\delta)$ (with bilipshitz constant 20). The tube meets the horosphere $z = 1$ in a region of area at most $NR\delta^2$. This area bound shows that we can fit at most $R\delta^{-2}N^{-1}$ disjoint tubes of this kind into $\tilde{T}$. The tubes fit together nicely, so this bound is sharp. To do the construction exactly, pick $\delta = \pi/N$. Since each tube has cross-sectional area δ^2, the total cross-sectional area of all these tubes is roughly R/N.

Now we make the additional assumption that N is a multiple of q. This assumption implies that each of our tubes in null-homologous in X. Some of the tubes stay in the central part of $\tilde{T}$ given by $x^2 + y^2 < (1/4)z^2R^2$. We call these inner tubes. Other tubes are disjoint from this central part, and we call them outer tubes. (A few tubes lie partly in the central part, and we throw them out.) We now compute the linking number in X of an inner tube with an outer tube. The core radial curve of the outer tube can be homotoped to the boundary of the Margulis tube T without crossing the inner tube, and it is homologous to $Mm + Nl$. The homology class $pm + ql$ bounds in the complement of T by assumption. Therefore, $(N/q)pm + Nl$ bounds in X. Subtracting such a boundary from our curve leaves a 1-cycle in the boundary of T homologous to $(M - Np/q)m$. Since the core circle of the inner tube winds N times around T in the positive direction, the linking number of m and the inner tube is N. Therefore, the linking number of our curve with the inner tube is $(M - Np/q)N$. We write this expression as $(M/N - p/q)N^2$. Since we assumed $\theta/(2\pi) = M/N$, we can rewrite this expression as $(\theta/(2\pi) - p/q)N^2$.

Using these linked tubes, we construct Lipschitz maps from X to S^2 with large Hopf invariants. Each of our tubes is bilipschitz to the product $S^1(Nr\epsilon) \times D^2(\delta)$. We pick a compactly supported degree 1 map ϕ_0 from $D^2(\delta)$ to the unit 2-sphere, with Lipschitz constant at most $10\delta^{-1}$. Then we extend this map to our tube by first projecting the tube to the disk and then using ϕ_0.

Now we consider three different maps from X to the unit 2-sphere, all with Lipschitz constant $C\delta^{-1}$. For F_1 use the tube map for each inner tube, and send all other points to the basepoint of S^2. For F_2 use the tube map for each outer tube, and send all other points to the base point of S^2. And for F_3, use the tube map for each inner and outer tube, and send all other points to the basepoint of S^2. Now the Hopf invariant is defined for all three maps F_1, F_2, and F_3. In terms of our linking data, we can directly compute $Hopf(F_3) - Hopf(F_1) - Hopf(F_2)$. This difference is given by twice the number of inner tubes times the number of outer tubes times the linking number of an inner tube with an outer tube. The number of inner tubes is at least $cR\delta^{-2}N^{-1}$. The number of outer tubes is lower bounded by the same expression. And the linking number is at least $|\theta/(2\pi) - p/q|N^2$. So the difference in Hopf invariants is at least $cR^2|\theta/(2\pi) - p/q|\delta^{-4} \geq c\epsilon^{-1/4}\delta^{-4}$. Therefore, the Hopf size of X is at least $c\epsilon^{-1/4}$.

So far, we have assumed that $\theta/(2\pi) = M/N$, where M and N are relatively prime integers and q divides N. This assumption is probably false, but it is easy to approximate $\theta/(2\pi)$ as closely as we like by such a fraction M/N. For a very good approximation, the Margulis tube $\tilde{T}$ is bilipshitz to the Margulis tube T' corresponding to a loxodromic isometry along a core geodesic of length ϵ with a twist of angle $(2\pi)M/N$. We construct all of the tubes and maps in T' as above and then pull them back to $\tilde{T}$ by our near-isometry. The resulting maps give the same estimate for the Hopf size. $\qquad\square$

To complement this proposition, we prove a different kind of inequality for hyperbolic 3-manifolds with short geodesics that are non-torsion in homology.

Proposition 4.2. *If a closed oriented hyperbolic 3-manifold X has a closed geodesic which is non-trivial in $H_1(X, \mathbb{Q})$ of length ϵ, then any surface with non-zero intersection number with γ must have both area and genus at least $c\epsilon^{-1/2}$.*

Proof. We consider as above the Margulis tube around the short geodesic γ, which has radius at least $R = c\epsilon^{-1/2}$. It contains thin tubes, each going around N times, with total cross-sectional area at least R/N. Any surface with a non-zero intersection number with γ meets each of these tubes at least N times, and so it has total area at least $N(R/N) = R > c\epsilon^{-1/2}$. By the Thurston simplex straightening argument, the genus of such a surface must be at least $c\epsilon^{-1/2}$ as well. $\qquad\square$

We define the spanning genus of a 3-dimensional simplicial complex M to be the smallest genus G so that $H_2(M, \mathbb{Q})$ is spanned by surfaces of genus at most G. We can rephrase the last proposition in this language: if X is a closed oriented hyperbolic 3-manifold with a closed geodesic of length ϵ which is not torsion in $H_1(X, \mathbb{Z})$, then the spanning genus of X is at least $c\epsilon^{-1/2}$.

In the last section, we proved that the Hopf size of a complex with N simplices is less than C^N. Similarly, we will prove that the spanning genus of a complex with N simplices is less than C^N.

Proposition 4.3. *Let M be a simplicial 3-complex with N simplices. Then the spanning genus of M is less than C^N.*

Proof. We consider the boundary map from the free vector space of 2-simplices of M to the free vector space of 1-simplices of M. The boundary map is given by a matrix M, with dimensions less than CN, and with each entry equal to ± 1 or 0. Moreover, since the boundary of each 2-simplex is only three 1-simplices, each column of the matrix has only three non-zero entries. Let r be the rank of the matrix M. Pick r 2-simplices Δ_i in X, so that $M(\Delta_i)$ gives a basis for the image of M. After renumbering the 2-simplices, we suppose that these are Δ_1 through Δ_r.

Now for every other 2-simplex Δ_j, we can express $M(\Delta_j)$ as a sum $a_{i,j}M(\Delta_i)$. (In this formula, the index j runs from $r+1$ to the last 2-simplex, and the index i runs from 1 to r.) Let I be the restriction of M to the span of the first r simplices. The linear map I is an isomorphism from $\mathbb{R}^r$ onto the image of M. This isomorphism has Hilbert-Schmidt norm less than $CN^{1/2}$, and it has determinant at least 1. Therefore, its inverse has norm at most C^N. For a fixed j, the sum $a_{i,j}\Delta_i$ is simply $I^{-1}(M(\Delta_j))$. Since $M(\Delta_j)$ has length $\sqrt{3}$, the vector $a_{i,j}$ has length less than C^N. In particular, each coefficient $a_{i,j}$ is less than C^N.

The coefficients $a_{i,j}$ don't have to be integers, but they are rational numbers with controlled denominators. The vectors $M(\Delta_j)$ are integral vectors in the image of

M. The vectors $M(\Delta_i)$ span a lattice L of integral vectors in the image of M. Let D be the index of the lattice L inside all of the integral vectors in the image of M. In particular, $DM(\Delta_j)$ lies in the integral span of the $M(\Delta_i)$, and so $Da_{i,j}$ is an integer for every i and j. Since the determinant of I is bounded by C^N, D is also bounded by C^N.

Now the integral 2-cycles $D\Delta_j - \sum_i Da_{i,j}\Delta_i$ span the kernel of the matrix M. Each of these 2-cycles has less than $NC^N C^N$ 2-simplices, and so it lies in the span of some surfaces with genus less than C^N. □

Over the course of the last proof, we have also bounded the size of the torsion subgroup of $H_1(M,\mathbb{Z})$. This subgroup is equal to the integral vectors in the image of M modulo the lattice L, and so D is the order of the torsion subgroup of $H_1(M,\mathbb{Z})$. As we showed above, this order is less than C^N. By an analogous argument, the torsion subgroup of $H^2(M,\mathbb{Z})$ has order less than C^N.

We can now give the proof of Theorem 2.

Theorem 2. *Let M be a closed oriented 3-dimensional manifold which can be triangulated by N simplices. Let X be a closed oriented hyperbolic 3-manifold with injectivity radius ϵ. If there is a map of non-zero degree from M to X, then ϵ is at least C^{-N}.*

Proof. Since X has injectivity radius ϵ, it must contain a closed geodesic of length less than 2ϵ. According to Propositions 4.1 and 4.2, either X has volume at least $c\epsilon^{-1/6}$, or $H_1(X,\mathbb{Z})$ has a torsion element with order at least $c\epsilon^{-1/6}$, or X has Hopf volume at least $c\epsilon^{-1/4}$, or X has spanning genus at least $c\epsilon^{-1/2}$.

If X has volume at least $c\epsilon^{-1/6}$, then according to Thurston's straightening theorem, N is greater than $c\epsilon^{-1/6}$, and we are done.

If $H_1(X,\mathbb{Z})$ has a torsion element of order T, greater than $c\epsilon^{-1/6}$, then we proceed as follows. Let α be this torsion class, and let α^* be the Poincare dual class in $H^2(X,\mathbb{Z})$. The element $f^*(\alpha^*)$ in $H^2(M,\mathbb{Z})$ must also be torsion with some order S dividing T. We claim that T divides the product of S with the degree of f. Let α_T be the image of α in $H_1(X,\mathbb{Z}_T)$. Let α_T^* be the Poincare dual class in $H^2(X,\mathbb{Z}_T)$. The pullback $Sf^*(\alpha_T^*)$ also vanishes. Since $H_0(X,\mathbb{Z})$ is free, $H^1(X,\mathbb{Z}_T)$ is equal to $Hom(H_1(X,\mathbb{Z}),\mathbb{Z}_T)$. In particular, we can choose a class a in $H^1(X,\mathbb{Z}_T)$ so that $a(\alpha) = a(\alpha_T) = 1$ modulo T. Therefore, the cup product $a \cup \alpha_T^*$ is equal to the fundamental cohomology class O_T in $H^3(X,\mathbb{Z}_T)$. We let $[M]_T$ denote the image of the fundamental homology class of M in $H_3(M,\mathbb{Z}_T)$. The degree of f modulo T is given by the pairing $f^*(O_T)([M]_T)$. Finally, S times the degree of f modulo T is given by $f^*(SO_T)([M]_T)$. But $f^*(SO_T)$ is equal to $f^*(a) \cup Sf^*(\alpha_T^*)$, which vanishes. Since the volume of X is at least on the order of 1, the degree of f is at most CN by Thurston's simplex straightening argument. Also, the number S is bounded by the order of the torsion subgroup of $H^2(M,\mathbb{Z})$, which is bounded by C^N. Therefore C^N must be greater than $c\epsilon^{-1/6}$, and we are done.

If the Hopf size of X is at least $c\epsilon^{-1/4}$, then Proposition 3.3 implies that $C^N \geq c\epsilon^{-1/4}$ and we are done.

If the spanning genus of X is at least $c\epsilon^{-1/2}$, we proceed as follows. Since the degree of f is non-zero, and since X obeys Poincare duality, the map f_* from $H_2(M,\mathbb{Q})$ to $H_2(X,\mathbb{Q})$ must be surjective. Therefore, the spanning genus of M must be at least $c\epsilon^{-1/2}$. But Proposition 4.3 tells us that the spanning genus of M is less than C^N, and we are done. □

The exponential bound in this theorem cannot be improved to a sub-exponential bound. To show this, we will construct closed oriented hyperbolic manifolds X which can be triangulated by N simplices and which contains closed geodesics of length less than c^{-N}, for a constant $c > 1$. Begin with a non-compact finite volume hyperbolic 3-manifold X_0 with a single cusp. We view X_0 as a manifold with boundary and we triangulate it. Let T be the restriction of our triangulation to the torus boundary. Next, we pick an Anosov diffeomorphism Ψ of the torus and a triangulation of the mapping cylinder of this diffeomorphism, which restricts to the triangulation T on each boundary component. Finally, we pick a curve c_0 in the boundary torus and a triangulation of a solid torus which restricts to T on the boundary and which Dehn fills the curve c_0. To form the 3-manifold X, we glue together X_0, N copies of the mapping cylinder, and one copy of the solid torus. (At each gluing, we glue one torus with triangulation T to another torus with the same triangulation by using the identity map.) The resulting 3-manifold can be triangulated with less than CN simplices. It is diffeomorphic to the Dehn filling of X_0 along the curve $\Psi^{-N}(c_0)$. If we fix any basis of $H_1(T^2)$, then for most choices of c_0, the coefficients of the homology class $\Psi_*^{-N}([c_0])$ grow exponentially with N. By Thurston's theory, almost all of these Dehn fillings are closed hyperbolic 3-manifolds, with uniformly bounded volume.

If the short geodesic in the core of the Dehn filling has length ϵ, then the radius of the horosphere cross-section of the Margulis tube must be roughly $\epsilon^{-1/2}$, and so there is a homologically non-trivial simple curve in the boundary of the Margulis tube of length less than $\epsilon^{-1/2}$ which bounds a disk in the Margulis tube. For large values of N, we have proved above that this length must be at least c^N, for a constant $c > 1$. Therefore, the length of the shortest geodesic is less than c^{-N}.

As a corollary of Theorem 2, we can give a new proof of the following result of Teruhiko Soma. Trying to understand Soma's result was the main motivation for the work in this paper.

Corollary. *(Soma, [6]) Given any closed oriented 3-manifold M, there are at most finitely many closed oriented hyperbolic 3-manifolds X with maps of non-zero degree from M to X.*

Proof. Suppose that M can be triangulated with N simplices. If there is a map of non-zero degree from M to X, then Theorem 2 tells us that the injectivity radius of X is at least C^{-N}. The estimate of Milnor and Thurston tells us that the volume of X is at most CN. By Cheeger's finiteness theorem, there are only finitely many hyperbolic manifolds obeying these bounds. □

5. Appendix 1: An example with large Hopf invariant

This exponential upper bound seemed very high to me at first. We will now construct examples to show that it cannot be improved to a sub-exponential bound.

Example 5.1. *There exists a triangulation of S^3 with N simplices so that after giving each simplex a unit Euclidean metric, the manifold has Hopf size greater than $exp(cN)$.*

Proof. We start with a torus T^2, equipped with a triangulation T and with a choice of basis for $H_1(T^2)$, called a and b, with intersection pairing of a and b equal to 1. Next we take the product $T^2 \times [0,1]$, and we Dehn fill both boundary components

to get S^3. More precisely, we Dehn fill the component $T^2 \times \{0\}$ in such a way that the homology class a bounds a disk in the new solid torus, and we Dehn fill the component $T^2 \times \{1\}$ in such a way that the homology class b bounds a disk in the new solid torus. We pick triangulations of the two solid tori extending T. (So far we have only constructed the simplest Heegaard splitting of S^3.)

Next we will construct a triangulation of the central cylinder $T^2 \times [0, 1]$ which restricts to the triangulation T on each boundary component. We pick an Anosov diffeomorphism Ψ of T^2. For instance, Ψ might act on homology by $\Psi_*(a) = 2a + b$ and $\Psi_*(b) = a + b$. We pick a triangulation of the mapping cylinder of Ψ which restricts to the triangulation T on each boundary component. Our triangulation of $T^2 \times [0, 1]$ consists of N copies of this triangulated mapping cylinder laid end to end, followed by N copies of the mirror image of the mapping cylinder. The whole triangulation involves less than CN simplices.

We will show that this triangulated 3-sphere contains two thick tubes with linking number at least $exp(cN)$. The first tube T_1 is localized near the middle of the long cylinder. Using the local coordinates for T^2, it is homologous to a. The second tube T_2 is the core of the Dehn filling of $T^2 \times \{0\}$. The linking number of T_1 with T_2 is equal to the pairing of $\Psi_*^N(b)$ with a, which grows exponentially with N.

Each tube T_i is diffeomorphic to a product $D^2 \times S^1$. We define a map f_i from T_i to S^2 by projecting to D^2 and then taking a degree 1 map from D^2 to S^2, mapping a neighborhood of the boundary of D^2 to the basepoint of S^2. We fix these maps so that they don't depend on N. In particular, they have uniformly bounded 2-dilation. Each map takes the boundary of T_i to the basepoint of S^2. We extend the map f_i to all of S^3 by mapping the complement of T_i to the basepoint of S^2. We construct a map f_3 from S^3 to S^2, whose restriction to each tube T_i is equal to f_i and which takes the rest of S^3 to the basepoint.

The maps f_i have Lipschitz constant at most C. The difference of Hopf invariants $Hopf(f_3) - Hopf(f_1) - Hopf(f_2)$ is given by twice the linking number of the two tubes, which is at least $(1 + c)^N$ for some universal $c > 0$. $\square$

The triangulated 3-manifold above has a large Hopf size, but it is homeomorphic to S^3. Therefore, any map from it to a surface of genus 2 is contractible and has Hopf invariant zero. Using the triangulation above to guide a surgical operation, we can construct manifolds M^3 which can be triangulated by N simplices and which admit maps to a surface of genus 2 with Hopf invariant at least $exp(cN)$.

Example 5.2. *We will construct closed oriented 3-manifolds M which can be triangulated by N simplices, and which admit maps to a surface of genus 2 with Hopf invariant at least $exp(cN)$.*

Proof. The manifold M is constructed by doing a kind of surgery on the above triangulation near the two tubes T_1 and T_2. For each tube, we do the following procedure. Identify T with $D^2 \times S^1$, and refine the triangulation so that the boundary of $D^2 \times S^1$ lies in the 2-skeleton. Next, we let Σ' be a surface of genus two with one boundary component. We cut out $D^2 \times S^1$ from our manifold and glue in $\Sigma' \times S^1$. The gluing map is the diffeomorphism from $\partial\Sigma' \times S^1$ to $\partial D^2 \times S^1$ given by the product of a diffeomorphism from $\partial\Sigma'$ to ∂D^2 and the identity map on S^1. Finally, we extend our triangulation to $\Sigma' \times S^1$. Since this operation occurs locally in a neighborhood of T, it adds only C extra simplices independent of N. Applying this procedure to both T_1 and T_2, we get our 3-manifold M.

Now we construct some maps from M to a closed surface of genus 2. Inside of M, we have two copies of $\Sigma' \times S^1$, which we added with our two surgeries. For each copy, we can construct a map in the following way. We map $\Sigma' \times S^1$ to Σ' by projecting to the first factor, and then compose with a degree 1 map from Σ' to a surface of genus 2, taking the boundary of Σ' to the base point of the target surface. The resulting map takes the boundary of $\Sigma' \times S^1$ to the base point of Σ, and so we can extend the map to all of M, mapping the rest of M to the basepoint of Σ. In this way, we construct two maps, called f_1 and f_2. We can also construct a third map f_3 by applying the above construction to both copies of $\Sigma' \times S^1$. As in the previous example, $\text{Hopf}(f_3) - \text{Hopf}(f_1) - \text{Hopf}(f_2)$ is greater than $exp(cN)$. $\square$

6. Appendix 2: The situation for non-orientable manifolds

Our proof of Soma's theorem does not extend to non-orientable manifolds, because intersection numbers, linking numbers, and the Hopf invariant are only defined modulo 2 in non-orientable manifolds. As Boileau and Wang discovered, the analogue of Soma's theorem is false for non-orientable manifolds.

Theorem. *(Boileau and Wang) There is a closed non-orientable 3-manifold M which admits degree 1 maps to infinitely many different closed oriented hyperbolic 3-manifolds X. (The degree is only defined modulo 2.)*

The proof of this result is a small modification argument of an argument of Boileau and Wang in [1]. For reference, we include a proof here.

Proof. The main idea of the proof is a clever choice of the domain M which is due to Wang. We begin with a finite volume hyperbolic manifold X_0 with a single cusp, which we view as a manifold with a torus boundary. Let us pick a basis for $H_1(\partial X_0)$, with elements a and b, so that the intersection product of a and b is equal to 1. We then perform a filling of X_0 roughly analogous to a Dehn filling but using a Mobius band in place of a disk. Let us make a precise construction. Let B denote the Mobius band. The boundary of $B \times S^1$ is a torus. We pick a diffeomorphism ϕ of the boundary of X_0 with the boundary of $B \times S^1$, so that $\phi(\partial B)$ is homologous to a and $\phi(S^1)$ is homologous to b. We define M to be the result of gluing X_0 to $B \times S^1$ with this diffeomorphism.

We will construct degree 1 maps from M to half of all the Dehn fillings of X_0. All but finitely many of these Dehn fillings admit hyperbolic metrics, and they include infinitely many different manifolds. For every pair (m, n) of relatively prime integers, let $X(m, n)$ denote the Dehn filling of X_0 along a curve homologous to $ma + nb$. The manifold $X(m, n)$ consists of the union of X_0 and a solid torus, glued along their boundaries. The map that we will construct from M to $X(m, n)$ takes X_0 to X_0 identically. The restriction of the identity map to the boundary gives a map from the boundary of $B \times S^1$ to the boundary of $D^2 \times S^1$.

We now investigate in what cases we can extend this map to a map from $B \times S^1$ to $D^2 \times S^1$. Since maps to D^2 automatically extend, it suffices to extend the map to S^1 from the boundary of $B \times S^1$ to the interior.

The homotopy classes of maps from the boundary of $B \times S^1$ to S^1 are simply $H^1(T^2)$. The cohomology class of our map is $n(\partial B)^* - m(S^1)^*$, where $(\partial B)^*$ is the cocycle that takes the value 1 on ∂B and the value 0 on S^1. Now B is homotopic to S^1. Let c denote a circle in B which is a deformation retract of B. Then $H^1(B \times S^1)$ is generated by c^* and $(S^1)^*$. The inclusion map of $\partial B \times S^1$ into $B \times S^1$ induces

a map on cohomology taking c^* to $2(\partial B)^*$ and $(S^1)^*$ to $(S^1)^*$. A given map from the boundary to S^1 extends to the interior if and only if the corresponding class in $H^1(\partial B \times S^1)$ lies in the image of $H^1(B \times S^1)$. This condition is met if n is even. Therefore, for every even n, there is a degree 1 map from the non-orientable 3-manifold M to the Dehn filling $X(m,n)$. □

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